УДК 517.954

BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS
WITH DIFFERENT BOUNDARY CONDITIONS

Yu. Yu. Krivosheeva, N. A. Slobozhanina

In technical applications, there are often nodes that can be subjected to thermal stress. Thermal load can lead to a change in a number of material properties, which will lead to device failure, damage or loss of important operational properties. To reduce the heat load, protective coatings can be applied to the surface. Thus, this reduces to the problem of the thermal conductivity of multilayer materials, the thermal conductivity of which changes abruptly. In this article, the first boundary-value problem was posed for the heat equation of a combined rod. An analytical solution to this problem was obtained in the form of a Fourier series, and the resulting series was evaluated. Based on the obtained solution and assessment, the process of heat distribution in the combined rod was simulated and graphs illustrating this process were obtained.

Key words: equation of heat conductivity, boundary value problem, method, variable separations, Fourier range, residual range.

Boundary value problems for the heat equation are one of the important sections of the equations of mathematical physics. In addition to the classical versions of problems for a homogeneous rod, cases where the rod is inhomogeneous are also interesting. Solving these problems may be of interest for areas related to multilayer structures. For example, such areas are medicine [1] or manufacturing [2] (manufacturing various multilayer parts for which they experience large temperature differences during operation).

Statement of the problem and construction of an analytical solution

Find the temperature distribution along a finite rod composed of two homogeneous rods touching at the point $x = l0$, with characteristics a and b, respectively. The lateral surface of the rods is thermally insulated, and the ends of the component rods are maintained at zero temperature. The temperature at the initial time in the component rods is different.

The mathematical model of the problem has the form: find the function $u(x,t)$, which is a solution to the equation

$$Lu \equiv \begin{cases} u_t - a^2 u_{xx} = 0, & 0 < x < l_0, \\ u_t - b^2 u_{xx} = 0, & l_0 < x < l, \end{cases}$$
$$0 < t < T,$$
$$0 < t < T, \qquad (1)$$

and satisfies the conditions

$$u(x,0) \equiv \begin{cases} \varphi_1(x), & 0 < x < l_0, \\ \varphi_2(x), & l_0 < x < l, \end{cases} \quad (2)$$

$$u(0,t) = u(l,t) = 0, \qquad 0 < t < T, \quad (3)$$

$$u(l_0 - 0, t) = u(l_0 + 0, t). \quad (4)$$

Using the method of separation of variables and the gluing condition, an analytical solution was obtained in the form of a series:

$$u(x,t) = \frac{(l - l_0)x \sum_{n=1}^{\infty} D_n(x,t) - l_0(l - x)\sum_{n=1}^{\infty} B_n(x,t)}{(x - l_0)l + l_0(l - l_0)(\sum_{n=1}^{\infty} C_n(x,t) - \sum_{n=1}^{\infty} A_n(x,t))} +$$
$$+ \frac{l_0(l - l_0)(\sum_{n=1}^{\infty} B_n(x,t) \sum_{n=1}^{\infty} C_n(x,t) - \sum_{n=1}^{\infty} A_n(x,t) \sum_{n=1}^{\infty} D_n(x,t))}{(x - l_0)l + l_0(l - l_0)(\sum_{n=1}^{\infty} C_n(x,t) - \sum_{n=1}^{\infty} A_n(x,t))}$$
$$(5)$$

where

$$\sum_{n=1}^{\infty} A_n(x,t) =$$
$$= \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{\pi n} e^{-\left(\frac{a\pi n}{l_0}\right)^2 t} \sin\left(\frac{\pi n}{l_0}x\right) \quad (6)$$

Krivosheeva Yuliana Yurievna (akinava.love@gmail.com),
student IV course of the Informatic Faculty;
Slobozhanina Natalia Aleksandrovna (slobogeanina@mail.ru),
associate professor of the Department of Foreign Languages and Russian as a Foreign Language
of the Samara University,
443086, Russia, Samara, Moskovskoye Shosse, 34.

$$\sum_{n=1}^{\infty} B_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{a\pi n}{l_0}\right)^2 t} \sin\left(\frac{\pi n}{l_0}x\right) =$$

$$= \frac{2}{l_0} \sum_{n=1}^{\infty} e^{-\left(\frac{a\pi n}{l_0}\right)^2 t} \sin\left(\frac{\pi n}{l_0}x\right) \int_0^{l_0} \varphi_1(s)\sin\left(\frac{\pi n}{l_0}s\right) ds \quad (7)$$

$$\sum_{n=1}^{\infty} C_n(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi n} e^{-\left(\frac{b\pi n}{l-l_0}\right)^2 t} \sin\left(\frac{\pi n(x-l_0)}{l-l_0}\right) \quad (8)$$

$$\sum_{n=1}^{\infty} D_n(x,t) = \sum_{n=1}^{\infty} D_n e^{-\left(\frac{b\pi n}{l-l_0}\right)^2 t} \sin\left(\frac{\pi n(x-l_0)}{l-l_0}\right) =$$

$$= \frac{2}{l-l_0} \sum_{n=1}^{\infty} e^{-\left(\frac{b\pi n}{l-l_0}\right)^2 t} \sin\left(\frac{\pi n(x-l_0)}{l-l_0}\right) \int_{l_0}^{l} \varphi_2(s)\sin\left(\frac{\pi n s}{l-l_0}\right) ds \quad (9)$$

Estimation of the remainder of the series and the number of terms

Let us estimate series (6) - (9) from above. As a result of the assessment, we obtain the following expressions:

$$R_N^A = \frac{l_0^2}{a^2\pi^3 N^2 t} e^{-\left(\frac{a\pi N}{l_0}\right)^2 t}$$

$$R_N^B = \frac{l_0^3}{a^2\pi^3 N^2 t} e^{-\left(\frac{a\pi N}{l_0}\right)^2 t}$$

$$R_N^C = \frac{(l-l_0)^2}{b^2\pi^3 N^2 t} e^{-\left(\frac{b\pi N}{l-l_0}\right)^2 t}$$

$$R_N^D = \frac{2(l-l_0)^2}{b^2\pi^3 N^2 t}\left(l_0 + \frac{l}{\pi}\right) e^{-\left(\frac{b\pi N}{l-l_0}\right)^2 t}$$

These estimates are needed in order to calculate how many terms must be summed to obtain a more accurate solution. The tables show the calculations for two given accuracy ε,

different t and the following parameter values: $a = 1; b = 0,3; l_0 = 1,7; l = 3$.

From the tables it follows that with this choice of parameters, the series converge quickly and, having summed up about 300 and 400 terms, we already achieve good accuracy ε = 0.001 and ε = 0.00001, respectively.

Charting

For plotting, a program was written in the C # programming language using the .NET Framework 4.5 framework in the Microsoft Visual Studio 2019 development environment. The program is a windowed application with fields for entering parameters a, b, l_0, l, time interval T, and buttons, when clicked, the graphs of the dependence $u(x)$ are drawn for a fixed value of t. The simulation results are presented in the figures 1-3.

It can be seen from the figures that a jump in temperature occurs at the point of discontinuity. In this case, the gluing condition is not violated, since the jump does not occur instantly. It can also be seen that as t approaches the end of a given time interval, the value of the function approaches zero, which indicates the establishment of the process.

Conclusion

So, an analytical solution to problem (1) - (4) was obtained and graphs of its visualization were constructed. Therefore, the tasks were completed, the goal of the work is achieved.

Table 1

The number of terms at $\varepsilon = 0,001$

	$t = 1$	$t = 0,1$	$t = 0,01$	$t = 0,001$	$t = 0,0001$
R_a	2	4	12	36	113
R_b	2	4	12	38	118
R_c	3	10	29	91	287
R_d	3	10	30	94	297

Table 2

The number of terms at $\varepsilon = 0,00001$

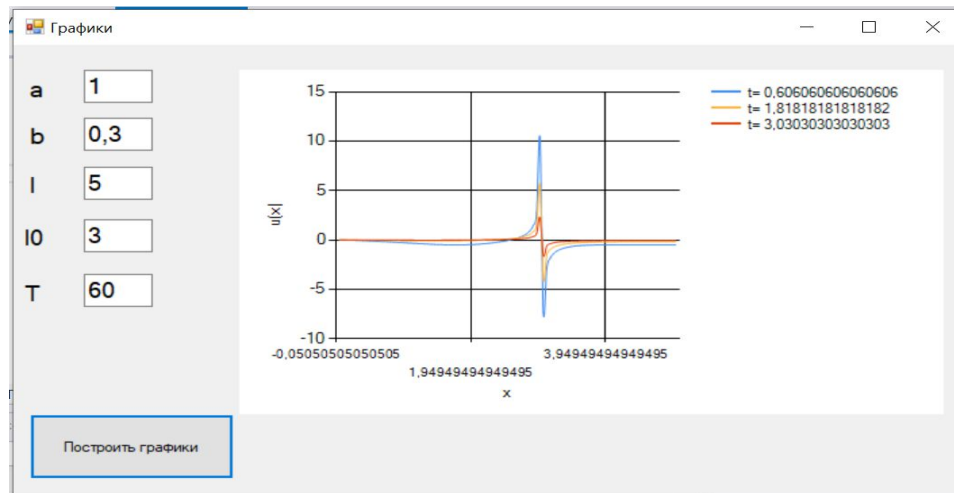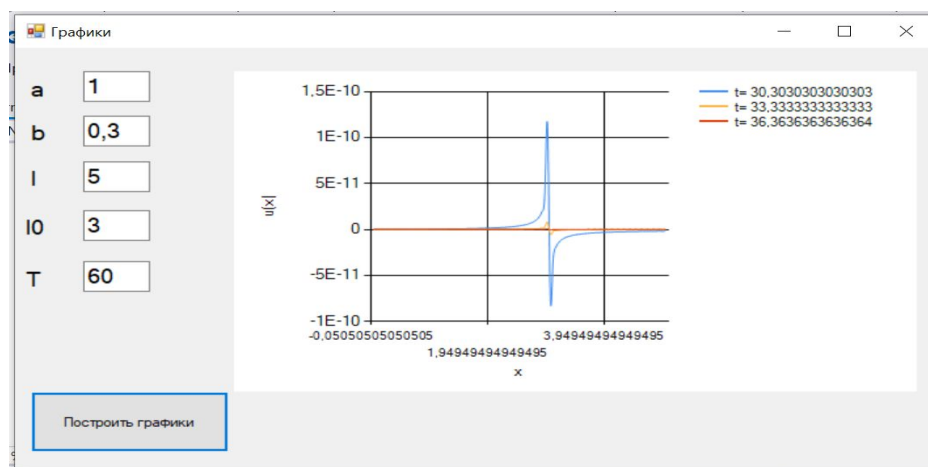|	$t = 1$	$t = 0,1$	$t = 0,01$	$t = 0,001$	$t = 0,0001$
R_a	2	5	16	50	156
R_b	2	6	16	51	160
R_c	4	13	40	126	397
R_d	5	13	41	129	405

Fig. 1. Graph of u (x) at the beginning of the time interval

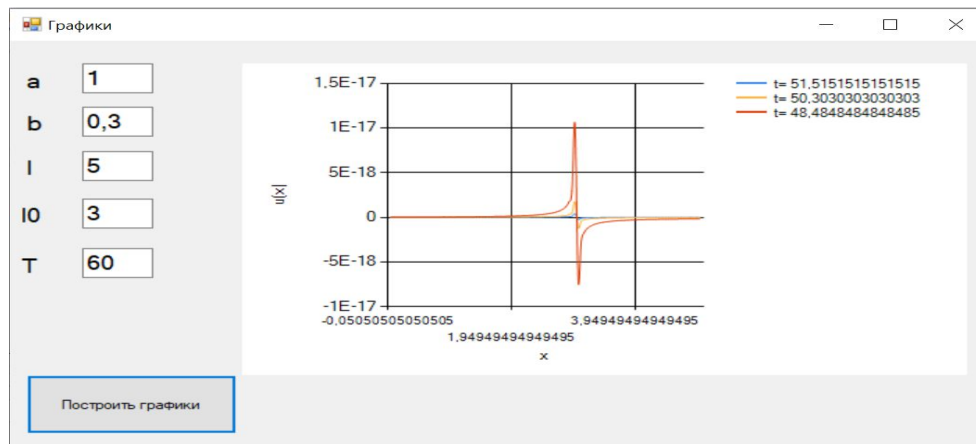**Fig. 2. Graph of u (x) at the middle of the time interval**

Fig. 3. Graph of u (x) at the ending of the time interval

References

1. Барун В. В., Иванов А. П. Световые и тепловые поля в многослойной ткани кожи при лазерном облучении // Оптика и спектроскопия. 2006. Т. 100. № 1. 149–157.

2. Танана В. П., Ершова А. А. О решении обратной граничной задачи для композитных материалов // Вестник удмуртского университета. 2018. Т. 28. № 4. С. 474–488.

3. Тихонов А. Н., Самарский А. А. Уравнения математической физики. М.: Наука, 1977. 735 с.

4. Сабитов К. Б. Уравнения математической физики. М.: Физматлит, 2013. 352с.

5. Штумпф С. А., Бахтин М. А. Математические методы компьютерных технологий в научных исследованиях. СПб: НИУ ИТМО, 2012. 148 с.

6. Документация по C#. URL: https://docs.microsoft.com/ru-ru/dotnet/csharp/ (дата обращения: 20.03.2020).

КРАЕВЫЕ ЗАДАЧИ ДЛЯ УРАВНЕНИЙ ПАРАБОЛИЧЕСКОГО ТИПА С РАЗЛИЧНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

Ю. Ю. Кривошеева, Н. А. Слобожанина

В технических приложениях часто встречаются узлы, которые могут подвергаться тепловой нагрузке. Тепловая нагрузка может привести к изменению ряда свойств материала, что приведет к выводу устройства из строя, его повреждению или утрате важных эксплуатационных свойств. Чтобы уменьшить тепловую нагрузку, можно наносить на поверхности защитные покрытия. Таким образом, это сводится к задаче о теплопроводности многослойных материалов, свойства теплопроводности которых меняются скачком. В данной работе была поставлена первая краевая задача для уравнения теплопроводности комбинированного стержня. Было получено аналитическое решение данной задачи в виде ряда Фурье и произведена оценка полученного ряда. На основе полученного решения и оценки было произведено моделирование процесса распространения тепла в комбинированном стержне и получены графики, иллюстрирующие этот процесс.

Ключевые слова: уравнение теплопроводности, краевая задача, метод разделения переменных, ряд Фурье, остаток ряда.

Кривошеева Юлиана Юрьевна (akinava.love@gmail.com),
студент IV курса факультета информатики;
Слобожанина Наталья Александровна (slobogeanina@mail.ru),
доцент кафедры иностранных языков и русского как иностранного Самарского университета,
443086, Россия, г. Самара, Московское шоссе, 34.